

Lorentz Covariant κ -Minkowski Spacetime

Ludwik Dąbrowski, Michał Godliński and Gherardo Piacitelli

SISSA, Via Bonomea 265, 34136, Trieste, Italy

June 30, 2010

Abstract

In recent years, different views on the interpretation of Lorentz covariance of non commuting coordinates have been discussed. By a general procedure, we construct the minimal canonical central covariantisation of the κ -Minkowski spacetime. Here, undeformed Lorentz covariance is implemented by unitary operators, in the presence of two dimensionful parameters. We then show that, though the usual κ -Minkowski spacetime is covariant under deformed (or twisted) Lorentz action, the resulting framework is equivalent to taking a non covariant restriction of the covariantised model. We conclude with some general comments on the approach of deformed covariance.

1 Introduction

The κ -Minkowski spacetime is defined by the commutation relations

$$[X^0, X^j] = iX^j, \quad j = 1, 2, 3, \quad (1a)$$

$$[X^j, X^k] = 0, \quad j, k = 1, 2, 3, \quad (1b)$$

among the four selfadjoint operators

$$(X^\mu) = (X^0, X^1, X^2, X^3),$$

to be interpreted as the coordinates of a noncommutative version of the usual Minkowski spacetime [1, 2]; see also [3, 4] for a detailed discussion. The name refers to the traditional notation κ for the inverse length scale; here we take natural units where $\kappa = 1$.

We do not consider commutation relations among coordinates and momenta, since we do not aim to a “more noncommutative” version of quantum mechanics. Indeed, energies involved at the Planck scale are so high that their description is expected to require a generalisation of quantum field theory. As an intermediate step, we are interested in generalizing the position space, not the quantum mechanical phase space. This should give a noncommutative replacement of the pointwise product of quantum fields, for a flat model (few processes of very high energy; see [5] for a discussion).

It was observed (see [6] and references therein), that the above relations can be generalised in the form

$$[X^\mu, X^\nu] = i(v^\mu X^\nu - v^\nu X^\mu), \quad (2)$$

where v is a fixed 4-vector in \mathbb{R}^4 . The standard timelike choice $v_{(0)} = (1, 0, 0, 0)$ reproduces precisely the usual κ -Minkowski spacetime (1).

In this model, covariance under Lorentz boosts and space translations has been sacrificed from the beginning, and replaced by κ -Poincaré symmetry, in the framework of quantum groups. The model has been under extensive investigation for almost 20 years (and still is), because of its nice mathematical features which make it a convenient framework where to test general ideas. See e.g. [7] for a recent review, focused on therecently proposed connections with Doubly Special Relativity.

From the point of view of spacetime quantisation, it has some inconvenient features: (i) the uncertainty relations are not sufficient to prevent an exceedingly high energy transfer to the geometric background by localisation, at least very close to the centre of space where the model is “nearly commutative”, (ii) on the contrary, at large scale, the model quickly becomes non commutative, so that e.g. it is not clear how to consistently formulate LHC physics already as far as 1mm from the centre of the κ -Minkowski, if $1/\kappa$ is of order of Planck length (see [3] for this and other estimates).

Here, we construct a new model which is the smallest extension of the κ -Minkowski spacetime, among those enjoying Lorentz covariance in the sense of Wigner. For this reason we call this model the *Lorentz covariant κ -Minkowski spacetime*. It is defined by the relations

$$[X^\mu, X^\nu] = i(V^\mu X^\nu - V^\nu X^\mu), \quad (3a)$$

$$[X^\mu, V^\nu] = 0, \quad (3b)$$

$$[V^\mu, V^\nu] = 0, \quad (3c)$$

$$V_\mu V^\mu = I, \quad (3d)$$

where I is identity operator and V is a purely vectorial quantity. If we took V to have the form $V^\mu = v^\mu I$ for some ordinary 4-vector v , we would fall back to timelike models in the class considered in [6]. In the spirit of [5], we propose instead to allow each V^μ to be a non trivial operator (not a multiple of the identity). In this case, the model admits a fully Lorentz covariant representation, namely such that there is a strongly continuous unitary representation U of the Lorentz group $\mathcal{L} = O(1, 3)$, fulfilling

$$U(\Lambda^{-1})X^\mu U(\Lambda) = \Lambda^\mu{}_\nu X^\nu, \quad (4a)$$

$$U(\Lambda^{-1})V^\mu U(\Lambda) = \Lambda^\mu{}_\nu V^\nu, \quad (4b)$$

which imply

$$U(\Lambda^{-1})[X^\mu, X^\nu]U(\Lambda) = i\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}(V^{\mu'}X^{\nu'} - V^{\nu'}X^{\mu'}). \quad (5)$$

Note that this is stronger than requiring simple form-covariance. For example, (5) cannot be obtained if the components of V are ordinary numbers, namely $V^\mu = v^\mu I$ as in (2). Indeed, unitary operators commute with multiples of the identity, so that $U(\Lambda^{-1})v^\mu X^\nu U(\Lambda) = v^\mu \Lambda^\nu_{\nu'} X^{\nu'}$ (no action on the index μ). For a covariant representation in the sense of (4), the (generalised) common eigenvalues v^μ of the pairwise commuting operators V^μ describe precisely the two-sheeted mass 1 hyperboloid.

Since star-products of symbols are a shadow of operator products, *full covariance* is precisely the appropriate notion for discussing the behaviour of star-products under Lorentz transformations of the symbols, in the spirit of Wigner theorem on quantum symmetries.

Note also that this provides yet another example (the first one being the DFR model [5]) where Lorentz symmetry is meaningful in the presence of an invariant characteristic length scale in a noncommutative setting, without deforming the Lorentz group nor its action. Even more, our covariantisation procedure gives a tool for constructing a rich family of such examples, where there are two invariant dimensionful parameters: the light speed c (here also set to one) and the inverse Planck length κ . In other words, a doubly special relativity framework does not require necessarily a deformation of the Lorentz group.

Obeying to the motto “no deformation without representation”, the covariant representation is explicitly described in the Mathematical Appendix, where the C^* -algebra of the model also is discussed.

The covariant κ -Minkowski spacetime arises as the *minimal canonical central covariantisation* of the usual κ -Minkowski model, presented in section 2. Our recipe for central covariantisation is minimal, in the sense that the family of (equivalence classes of) irreducible representations cannot be pruned any further without destroying the possibility of building up a covariant representation.

It is clear that the κ -Minkowski relations (1) can be obtained by supplementing relations (3) with the constraint

$$V^0 = I, \tag{6}$$

which, by (3d,3c), entails $V^1 = V^2 = V^3 = 0$. Condition (6) may be regarded as a criterion for selecting the usual κ -Minkowski out of the fully covariant model, as a subrepresentation of the latter. In this sense the κ -Minkowski model is a reduction of the fully covariant model, which reproduces the very same relationship [8] between the full DFR model [5] and the reduced DFR model (a member of the family of the so called “canonical quantum spacetimes”).

Since the class of representations (or equivalently of localisation states, according to the GNS theorem) selected by (6) is not invariant, the resulting theory, though formally covariant, breaks the relativity principle, in the sense already discussed in [9]: the selection of admissible representations and states is defined with respect to some preferred observer in his/her special Lorentz frame.

To complete the picture, we show that κ -Minkowski spacetime admits an approach based on “deformed covariance”, which parallels the “twisted covariance” of [10, 11, 12]. This provides a deformation operator (“twist”) realising the

star product, in the case of timelike choices of v in (2) (for a discussion of the lightlike case in relation with the classical r-matrices, see [6]). However, the resulting formalism is found plainly equivalent to rejecting all the representations with $V^0 \neq I$.

We finally draw some conclusions, in particular concerning the rôle of “twists” in spacetime quantisation, and the (lack of) physical motivations for rejecting otherwise admissible localisation states.

2 Minimal Canonical Central Covariantisation

Assume that we have a finitely generated model of a flat, noncommutative spacetime described in terms of the selfadjoint operators X^N , $N = 0, 1, 2 \dots \bar{N}$, where $3 \leq \bar{N} < \infty$. The first four operators X^0, \dots, X^3 are to be interpreted as the non commutative coordinates of a 4-event.

In order to get compact equations, we set $(g^{MN}) = \text{diag}(1, -1, -1 \dots, -1)$ as a $(\bar{N} + 1) \times (\bar{N} + 1)$ matrix, and we extend the usual conventions about implicit summation, raising and lowering to the full set of indices; this is a purely formal convention, with no underlying interpretation, and no loss of generality. When we use Greek symbols μ, ν, \dots we understand dummy indices running in the set $\{0, 1, 2, 3\}$; Latin dummy indices m, n, \dots run in $\{4, 5, \dots, \bar{N}\}$, while capital Latin dummy indices M, N, \dots run in the full set $\{0, 1, 2, \dots, \bar{N}\}$.

This done, we assume the commutation relations to have the form

$$[X^M, X^N] = i\theta^{MN}{}_R X^R \quad (7)$$

for a given real tensor θ , antisymmetric in the two upper indices, and fulfilling the constraints imposed by the Jacobi identity; i is the imaginary unit.

The above framework incorporates a large class of models, among which we mention

1. The canonical quantum spacetime, where $\bar{N} = 4$, $X^4 = I$, and

$$\theta^{MN}{}_\rho = 0, \quad \theta^{Mn}{}_4 = 0.$$

If we choose $\sigma_{(0)}^{\mu\nu} = \theta^{\mu\nu}{}_4$ to be the standard symplectic matrix (or any of its Lorentz transforms), we obtain the reduced DFR model [5].

2. The κ -Minkowski spacetime, where $\bar{N} = 3$, there only are Greek dummy indices, and

$$\theta^{\mu\nu}{}_\rho = g^\mu{}_\rho v_{(0)}{}^\nu - g^\nu{}_\rho v_{(0)}{}^\mu$$

with $v_{(0)} = (1, 0, 0, 0)$.

We now wish to consider the effect of Lorentz transformations on the relations among the physical coordinates. We take the point of view that the additional operators $X^4, \dots, X^{\bar{N}}$ correspond to inner degrees of freedom of the

noncommutative spacetime (a fixed background). Hence they are unaffected by a Lorentz transformation. It is clear that, defining new operators X' by

$$\begin{aligned} X'^{\mu} &= \Lambda^{\mu}_{\nu} X^{\nu}, \\ X'^n &= X^n, \end{aligned}$$

they fulfil

$$[X'^M, X'^N] = i\theta'^{MN}{}_R X'^R,$$

where θ' is obtained by transforming the Lorentz (Greek) values only of the indices of θ .

We may regard θ , describing the commutator of the initial non covariant model, as a collection of Lorentz tensors, parametrized by the non Lorentz (lower case Latin) values of the indices. In particular, fixing two indices to take non Lorentz values, we have vectors with index μ , parametrised by m, n :

$$\theta^{\mu m}{}_n, \theta^{m\mu}{}_n, \theta^{mn}{}_{\mu};$$

the second is redundant because of antisymmetry, so that we have the following independent Lorentz invariants, labeled by n, m :

$$\begin{aligned} \phi_1(\theta) &= \theta^{\mu m}{}_n \theta_{\mu}{}^m{}_n, \\ \phi_2(\theta) &= \theta^{mn\mu} \theta^{mn}{}_{\mu}, \end{aligned}$$

(no summation over Latin indices).

With two Lorentz indices, we have

$$\theta^{\mu\nu}{}_n, \quad \theta^{\mu n}{}_{\nu}$$

where again we used antisymmetry in the upper index to discard the redundant one. For the first of them, we have the following independent Lorentz invariants

$$\begin{aligned} \phi_3(\theta) &= \theta^{\mu\nu}{}_n \theta_{\mu\nu n}, \\ \phi_4(\theta) &= (\theta^{\mu\nu}{}_n (*\theta_n)_{\mu\nu})^2, \end{aligned}$$

(no summation over Latin indices),

where $*\theta_n$ stands for the Hodge dual of $\theta^{\mu\nu}{}_n$ in the indices μ, ν , holding n fixed. We will not list all the invariants of $\theta^{\mu n}{}_{\nu}$ (which is not symmetric) and of the only tensor with three Lorentz indices $\theta^{\mu\nu}{}_{\rho}$, not to burden the presentation, and because we will not need them here.

Hence, we may define a central covariantisation of the initial model by simply turning the numbers $\theta^{MN}{}_R$ into central operators $\Theta^{MN}{}_R$, setting

$$[X^M, X^N] = i\Theta^{MN}{}_R X^R, \tag{8a}$$

$$[X^M, \Theta^{MN}{}_R] = 0, \tag{8b}$$

$$[\Theta^{MN}{}_R, \Theta^{M'N'}{}_{R'}] = 0. \tag{8c}$$

The above certainly contains the given initial model as a subrepresentation. However, this would add way too many new representations to the initial model; the only remaining dependence on the initial model is on the number of generators, and in a sense we may regard it as the maximal central covariantisation. We want a covariant model that fulfils two natural requirements, namely, (i) it contains the initial model as a subrepresentation, and (ii) it is the smallest possible covariant extension of the initial model. We obtain this by adding all invariant constraints on the admissible representations, which are fulfilled by the initial model:

$$\phi_j(\Theta) = \phi_j(\theta)I, \quad j = 1, 2, 3, \dots;$$

more explicitly,

$$\Theta^{\mu m}{}_n \Theta_\mu{}^m{}_n = \theta^{\mu m}{}_n \theta_\mu{}^m{}_n I, \quad (8d)$$

$$\Theta^{mn\mu} \Theta_{mn}{}_\mu = \theta^{mn\mu} \theta_{mn}{}_\mu I, \quad (8e)$$

$$\Theta^{\mu\nu}{}_n \Theta_{\mu\nu}{}_n = \theta^{\mu\nu}{}_n \theta_{\mu\nu}{}_n I, \quad (8f)$$

$$(\Theta^{\mu\nu}{}_n (*\Theta_n)_{\mu\nu})^2 = (\theta^{\mu\nu}{}_n (*\theta_n)_{\mu\nu})^2 I, \quad (8g)$$

$$\dots \quad (8h)$$

(no summation over Latin indices),

where the ellipsis stand for the analogous equations corresponding to the invariants which we did not write down explicitly.

Altogether, the relations (8) define the minimal central covariantization of the relations (7). The above strategy is canonical, in the sense that it does not depend on any additional arbitrary choice, once the choice of the initial relations (7) is made.

Let us see two applications of the above mentioned general strategy.

1. In the reduced DFR model, θ contains the following Lorentz tensors,

$$\begin{aligned} \theta^{\mu 4}{}_4, \quad & \text{(one Lorentz index),} \\ \theta^{\mu\nu}{}_4, \quad \theta^{\mu 4}{}_\nu \quad & \text{(two Lorentz indices),} \\ \theta^{\mu\nu}{}_\rho \quad & \text{(three Lorentz indices).} \end{aligned}$$

The only one which is not identically vanishing is $\theta^{\mu\nu}{}_4 = \sigma_{(0)}^{\mu\nu}$. Two independent invariants of this tensor are

$$\sigma_{(0)\mu\nu} \sigma_{(0)}^{\mu\nu} = 0, \quad \frac{1}{16} (\sigma_{(0)\mu\nu} (*\sigma_{(0)})^{\mu\nu})^2 = 1.$$

Hence the covariantized model is defined by the Eqs. (8a-8c), complemented with the constraints

$$\begin{aligned} \Theta^{\mu 4}{}_N &= 0, \\ \Theta^{\mu\nu}{}_\rho &= 0, \\ \Theta_{\mu\nu}{}_4 \Theta^{\mu\nu}{}_4 &= 0, \\ \left(\frac{1}{4} \Theta_{\mu\nu}{}_4 (*\Theta_4)^{\mu\nu} \right)^2 &= X^4. \end{aligned}$$

With the identifications

$$q^\mu = X^\mu, \quad Q^{\mu\nu} = \Theta^{\mu\nu}_4, \quad I = X^4,$$

we recognise the commutation relations of the DFR model [5].

2. The κ -Minkowski spacetime, where $\bar{N} = 3$, dummy indices take only Greek values, and

$$\theta^{\mu\nu}{}_\rho = g^\mu{}_\rho v_{(0)}{}^\nu - g^\nu{}_\rho v_{(0)}{}^\mu$$

with $v_{(0)} = (1, 0, 0, 0)$. In this case the tensor θ has no traceless component and there only is the vectorial part. The tensor θ is uniquely associated to $v_{(0)}$, since $v_{(0)}{}^\mu = (1/3)\theta^{\rho\mu}{}_\rho$. So we may forget θ and make our statements directly in terms of $v_{(0)}$, whose only invariant is $\phi(v_{(0)}) = v_{(0)}{}_\mu v_{(0)}{}^\mu = 1$. Then we turn $v_{(0)}$ into a vector V with operator entries. A centrally covariantized model (not the maximal one, since tensors with traceless components are implicitly ruled out) is given by (3a,3b,3c). We restrict to the minimal covariantization through the relation $\phi(V) = \phi(v_{(0)})I$, namely (3d).

We show in the Appendix that there exists a covariant representation: selfadjoint operators X^μ, V^μ and a unitary representation U of the Lorentz group, fulfilling (3) and

$$\begin{aligned} U(\Lambda^{-1})X^\mu U(\Lambda) &= \Lambda^\mu{}_\nu X^\nu, \\ U(\Lambda^{-1})V^\mu U(\Lambda) &= \Lambda^\mu{}_\nu V^\nu. \end{aligned}$$

3 Deformed Covariance

Another, apparently unrelated, approach to covariance is described in this section; it was proposed independently by [10, 11], to obtain quantum group deformations of the Lorentz group, in the case of the canonical quantum spacetime (see also [13]).

To a certain extent, the basic argument is of considerable generality, and we will apply it to the κ -Minkowski case.

Let

$$m(f \otimes g) = fg$$

be the usual (commutative) pointwise multiplications of functions. Let

$$(\alpha_\Lambda f)(x) = f(\Lambda^{-1}x)$$

be the usual action of the Lorentz group, and $\alpha_\Lambda^{(2)} = \alpha_\Lambda \otimes \alpha_\Lambda$ so that

$$\alpha_\Lambda^{(2)}(f \otimes g)(x, y) = f(\Lambda^{-1}x)g(\Lambda^{-1}y).$$

The usual multiplication m intertwines the actions α_Λ and $\alpha_\Lambda^{(2)}$:

$$m \circ \alpha_\Lambda^{(2)} = \alpha_\Lambda \circ m. \tag{9}$$

Now, assume that an associative product \star_0 of ordinary functions of \mathbb{R}^4 is given. Assume also that we are able to find an *invertible* operator F such that

$$f \star_0 g = m \circ F(f \otimes g). \quad (10)$$

Once such an F is found to exist, one may deform $\alpha_A^{(2)}$ into

$$\tilde{\alpha}_A^{(2)} := F^{-1} \circ (\alpha_A \otimes \alpha_A) \circ F.$$

Let us now introduce the notation $m_{\star_0}(f \otimes g) = f \star_0 g$. We observe that

$$m_{\star_0} \circ \tilde{\alpha}_A^{(2)} = m \circ \alpha_A^{(2)} \circ F,$$

while

$$\alpha_A \circ m_{\star_0} = \alpha_A \circ m \circ F,$$

Since F is invertible, equality between the right hand sides of the two equations above is equivalent to (9). It follows that the left hand sides of the above are equal, and m_{\star_0} fulfils “deformed covariance”:

$$m_{\star_0} \circ \tilde{\alpha}_A^{(2)} = \alpha_A \circ m_{\star_0}, \quad (11)$$

which is a deformation of (9).

This trick works whenever an operator such as F exists. Its precise form is not relevant in the game, and actually when it exists it is highly non unique. F is called a twist in the case of the canonical quantum spacetime, since it reproduces (in momentum space) the twisted convolution product according to the terminology of [14].

We may apply these ideas to the case of the κ -Minkowski spacetime as well, instead of covariantizing it as described in the preceding sections. We keep up with the original, non covariant coordinates X^μ , fulfilling (1) (as operators on some Hilbert space \mathfrak{H}). Under the quantization prescription *à la Weyl*

$$f(X) = \frac{1}{(2\pi)^2} \int d\alpha \hat{f}(\alpha) e^{i\alpha_\mu X^\mu} \quad (12)$$

(as an operator on \mathfrak{H}), we define a star product through

$$(f \star_0 g)(X) = f(X)g(X), \quad (13)$$

where f, g are ordinary functions of \mathbb{R}^4 , often called (Weyl) symbols in the theory of pseudodifferential operators. If properly treated, the symbolic calculus can be used as an equivalent replacement of the underlying operator algebra.

We finally prove the existence of a suitable deformation operator F in the case of κ -Minkowski spacetime. We refrain from giving a complete computation, since the functional form of F is irrelevant.

Note that we prefer to call F the *deformation operator*, since in this case it is not associated with a twisted convolution: indeed, contrary to the case

of canonical commutation relations, the appropriate class of Weyl operators is closed under operator products [4]. The relations with twists in the sense of Drinfel'd and the Hopf-theoretical approach will be discussed elsewhere.

Both in the present case and in the case of the twisted product, it is easier to work in Fourier space and look for an invertible operator T such that

$$(c \circ T)(\hat{f} \otimes \hat{g}) = \hat{f} *_0 \hat{g}, \quad (14)$$

where $*_0$ is deformed product in Fourier space, and c is ordinary convolution. Then the desired F is obtained by

$$\widehat{Ff \otimes g} = T(\hat{f} \otimes \hat{g}).$$

Moreover, since the algebra of relations in $3+1$ dimensions is a central extension of the algebra in $1+1$ dimensions ([3]; see Mathematical Appendix, eqn.s (16)), it is sufficient to prove existence of T in the latter case.

Let

$$w(\alpha, \alpha') = \frac{\alpha(e^{\alpha'} - 1)}{\alpha'(e^\alpha - 1)};$$

we recall (from [4]; but see from¹) that in $1+1$ dimensions

$$(\hat{f} *_0 \hat{g})(\alpha, \beta) = \int d\alpha' d\beta' w(\alpha - \alpha', \alpha) \hat{f}(\alpha', \beta') \hat{g}(\alpha - \alpha', w(\alpha - \alpha', \alpha)\beta - w(\alpha' - \alpha, \alpha')\beta').$$

Then

$$(T \hat{f} \otimes \hat{g})(\alpha, \beta, \alpha', \beta') = w(\alpha', \alpha' + \alpha) \hat{f}(\alpha, \beta) \hat{g}(\alpha', (w(\alpha', \alpha' + \alpha) - w(\alpha', \alpha))\beta - w(\alpha', \alpha)\beta')$$

fulfils (14), where $*_0$ is now the deformed κ -Minkowski product in Fourier space for $1+1$ dimensions. Moreover,

$$\hat{f}(\alpha, \beta) \hat{g}(\alpha', \beta') = w(\alpha' + \alpha, \alpha')(T \hat{f} \otimes \hat{g})(\alpha, \beta, \alpha', (1 - w(\alpha' + \alpha, \alpha))\beta - \beta'),$$

which gives invertibility of T .

4 Deformed Covariant κ -Minkowski as a Non Invariant Reduction

Now, we come back to the fully covariant model. In this case the presence of a non trivial centre forbids to define a star product through a quantization prescription of the kind of (12) for symbols only depending on x , since then a requirement of the form (13) would be inconsistent: indeed the right hand side would fail to be an object of the form of the left hand side. To say it differently,

¹Here, we consider the star product defined with the “symmetric” Weyl prescription for Weyl operators, not to be confused with the “time first” and “space first” prescriptions. The symmetric form of the Weyl operators was proposed in [15], based on the integration of the BCH formula of [16]. In connection with this, see also [17].

there is the necessity of considering a v dependence to account for the centre of the algebra. This problem has been thoroughly discussed in [5] in a different context (see also the less technical [18]), so that we only shortly describe the analogous solution in our case.

We consider a more general class of symbols, namely functions $f = f(v, x)$ of $H \times \mathbb{R}^4$, where the two-sheeted, mass 1 hyperboloid H arises as the set of common generalised eigenvalues (joint spectrum) of the pairwise commuting operators V^μ (see the appendix). For such symbols, we define a quantization in two steps. First, for each x fixed, we replace v by V in the usual sense of functions of pairwise commuting operators, so to obtain an operator valued function

$$x \mapsto f(V, x).$$

Next we consider the quantization

$$f(V, X) = \frac{1}{(2\pi)^2} \int d\alpha \hat{f}(V, \alpha) e^{i\alpha_\mu X^\mu},$$

where

$$\hat{f}(V, \alpha) = \frac{1}{(2\pi)^2} \int dx \hat{f}(V, x) e^{-i\alpha_\mu x^\mu}.$$

The above prescription is unambiguous, since $[X^\mu, V^\nu] = 0$. Now,

$$(f \star g)(V, X) = f(V, X)g(V, X)$$

gives a consistent definition of the star product. The \ast -algebra of such generalised symbols is sufficient to fully describe the operator algebra arising from quantization.

Note that, for each $v \in H$ fixed, we may define a deformed product “at v ” by

$$f(v, \cdot) \star_v g(v, \cdot) = (f \star g)(v, \cdot),$$

which is meaningful as a product of reduced symbols depending on x only. Every such star product is precisely the star product which would be defined by

$$f(v, X_{(v)})g(v, X_{(v)}) = (f(v, \cdot) \star_v g(v, \cdot))(X_{(v)}),$$

if $X_{(v)}$ were the coordinates defined by (17) of the appendix, and $f(v, \cdot), g(v, \cdot)$ are thought of as functions of x only, parametric in v .

In particular, $\star_{v(0)}$ is precisely the star product \star_0 of the usual κ -Minkowski spacetime, introduced in the preceding section.

We may define an action of the Lorentz group on the algebra of generalised symbols, by

$$(\tau_\Lambda f)(v, x) = f(\Lambda^{-1}v, \Lambda^{-1}x).$$

If X, V, U is a covariant representation fulfilling (3,4), then we find

$$(\tau_\Lambda f)(V, X) = U(\Lambda)f(V, X)U(\Lambda)^{-1}.$$

In particular,

$$(\tau_\Lambda f) \star (\tau_\Lambda g)(v, x) = \tau_\Lambda(f \star g)(v, x).$$

Now, we observe that

$$(\tau_\Lambda f) \star (\tau_\Lambda g)(v, x) = f(\Lambda^{-1}v, \Lambda^{-1}\cdot) \star_v g(\Lambda^{-1}v, \Lambda^{-1}\cdot),$$

so that in particular if we take generalised symbols f, g which are constant in v for every x , and we evaluate the above at $v = \Lambda v_{(0)}$, we get

$$(\tau_\Lambda f) \star (\tau_\Lambda g)(\Lambda v_{(0)}, x) = ((\alpha_\Lambda f) \star_{\Lambda v_{(0)}} (\alpha_\Lambda g))(x),$$

where α_Λ only affects the x dependence of symbols, defined in the preceding section. By comparison with (11) we recognise that

$$(\alpha_\Lambda f) \star_{\Lambda v_{(0)}} (\alpha_\Lambda g) = m_{\star_0} \circ \tilde{\alpha}_\Lambda^{(2)}(f \otimes g); \quad (15)$$

so that the right hand side may be regarded merely as an alternative notation for the left hand side.

In other words, deformed covariance (deformed Lorentz action, same product) is equivalent to usual (form) covariance (usual Lorentz action, Lorentz transformed product), in complete analogy with the analysis of [8].

Since this situation is not special to this model, but a consequence of central covariantization which is always possible in principle, in a sense even the existence of F is not essential, since in the end the only rôle it plays is that of an equivalent notation for $\star_{\Lambda v_{(0)}}$, which is always meaningful.

We now give a physical interpretation of the above formal developments. Localisation states on the algebra are linear functionals

$$\omega(f) = \int_{H \times \mathbb{R}^4} dv \, dx \, \rho(v, x) f(v, x);$$

it is rather difficult to express in terms of conditions on the kernel ρ the fundamental properties of a state, in particular positivity:

$$\omega(\bar{f} \star f) \geq 0.$$

However we may select a special class of states by requiring that, when restricted to functions of v only, they are atomic measures concentrated on $v_{(0)}$, namely states of the form

$$\omega(f) = \int dx \, \rho'(x) f(v_{(0)}, x)$$

for some ρ' . If we took as a fundamental assumption of the theory that the only admissible localisation states were precisely those appearing in the above form to some given observer (conventionally called the privileged one), he only would “see” the usual κ -Minkowski spacetime, and all the rest of the structure would remain hidden to him. He would be naturally led to use symbols not depending on v , and the product $\star_0 = \star_{v_{(0)}}$.

Another observer, connected to the privileged one by Λ , only could access the states obtained by the pull back action of the Lorentz group on symbols, so she only could see symbols evaluated at $v = \Lambda v_{(0)}$. She would use the product \star_v , which can be written equivalently using deformed covariance.

Hence the formalism of deformed covariance is equivalent to work with a fully covariant model, up to arbitrarily dismissing a huge family of otherwise admissible localisation states. It is this very last step which destroys covariance.

5 Conclusions

Whenever a set of more or less physically motivated commutation relations is given, which define a candidate for noncommutative spacetime, then full Lorentz symmetry is not an issue. We have seen that there is a standard strategy (covariantization) leading to a fully covariant model. Of course, in principle there may be other — more complex — strategies which do not end up with central extensions. Anyhow, whatever strategy is taken, if physical motivations really are motivating one should check whether they survive the covariantization or not. In the first case, all is well that ends well; in the second case, motivations might be so strong to make us consider the breakdown of the relativity of observers as an acceptable cost. Or they might not: in the absence of strong physical motivations a non covariant choice has no evident payoff.

We noted that the approach of deformed covariance (in the spirit of twisted covariance) is possible whenever a certain invertible deformation operator F exists; its exact form is irrelevant, existence is all one needs. Indeed, we might go one step further: neither existence of F is necessary, F can be postulated. Then everything seems to go fine, since all equations can be given sense *a posteriori* in terms of the corresponding covariantization, even if the deformation operator eventually turns out not to exist.

In the end, the only requirement which is made on the deformation operator F is to reproduce the deformed product when $Ff \otimes g$ is restricted to the diagonal. Hence F is highly non unique, and undetermined off the diagonal. Any physical consequence deriving from the off diagonal behaviour of F would require some argument motivating the particular choice of F . Otherwise, for any formal development which does not depend on the particular choice of the off diagonal behaviour of F , one is entitled to conjecture that it can be reproduced in the covariantized setting as well.

This seems to suggest that the deformation operator might contain no additional physically interesting information with respect to the underlying covariant algebra.

One substantial difference between the two approaches to covariance appears to be the possibility of considering commutation relations depending on the spacetime event, in connection with gauge theories. This is sometimes claimed to only be possible in the formalism of deformed covariance, at least in the case of twists. However, there are two aspects which should be thoroughly discussed.

Indeed, (i) we should agree on the meaning of “possible”. If one takes ex-

istence of operators fulfilling regular commutation relations as fundamental, it is not clear that locally twisted commutation relations among the coordinates admit any representation by selfadjoint operators; this is not an idle problem, since the representations determine the full algebraic content (the universal C^* -algebra) of the theory. On the contrary, if deformed products are taken as fundamental (namely disregarding representations), the language of operators and their formal commutation relations should be considered as unavailable.

Last but not least, (ii) when speaking of local twists we implicitly give a meaning to infinitely small (classical) points of classical spacetime. This appears to be in plain contrast with the fundamental idea of noncommutative geometry, where the concept of “point” is dismissed as a fundamental one. Classical points may be given a meaning only as derived concepts; yet the family of classical points (which may well be empty as in the DFR model) can be sufficient to realise the whole spectrum of the localisation algebra in the commutative case only (Gel’fand’s theorem).

Mathematical Appendix

We show here that there is a (essentially unique) covariant representation of (3), namely selfadjoint operators X^μ, V^μ and a strongly continuous unitary representation U of the Lorentz group, fulfilling (3,4).

We first focus on irreducible representations: by Schur’s lemma, all central quantities must be multiples of the identity in an irreducible representation, so that we must have $V^\mu = v^\mu I$ for some real vector $v \in H = H_- \cup H_+$, the mass 1 hyperboloid with connected components

$$H_\pm = \{v \in \mathbb{R}^4 : \pm v^0 > 0, v_\mu v^\mu = 1\}.$$

Here H plays the rôle analogous to that of the manifold Σ in [5]. In this case, we say we face an irreducible representation belonging to v , for short. Irreducible representations belonging to different vectors are clearly inequivalent. Let now v, v' be any two such vectors; there always is some Lorentz matrix Λ such that $\Lambda v = v'$. Given an irreducible representation X belonging to v , $X' = \Lambda X$ also is a representation, which belongs to v' . Since Λ is invertible, X' also is irreducible. Taking linear combinations commute with the adjoint action of unitary operators; hence this correspondence sends equivalence classes belonging to v into equivalence classes belonging to v' , and vice versa. Thus, it suffices to classify irreducible representations belonging to one only $v_{(0)}$, e.g.

$$v_{(0)} = (1, 0, 0, 0).$$

We now may take profit from the fact [3] that all irreducible representations belonging to $v_{(0)}$ appear in the disintegration of the following, universal representation

$$\begin{aligned} X_{(v_{(0)})}^0 &= I \otimes P, \\ X_{(v_{(0)})}^j &= C^j \otimes e^{-Q} \end{aligned}$$

acting on the Hilbert space $\mathfrak{H}_0 = \mathfrak{K} \otimes L^2(\mathbb{R})$, where P, Q are the usual Schrödinger operators on the line, fulfilling $[P, Q] = -iI$; while the C^j s are pairwise commuting bounded selfadjoint operators on some Hilbert space \mathfrak{K} with joint spectrum² $S^2 \cup \{0\} \subset \mathbb{R}^3$. It follows that $E = \sum_j C^{j^2}$ is an orthogonal projection, such that $R^2 := \sum_j X^{j^2} = E \otimes e^{-2Q}$; hence this quantization only involves time and the distance from the origin, while angle variables remain classical. In this sense the quantization is radial; it may be obtained equivalently from the relations

$$[T, R] = iR, \quad (16a)$$

$$[T, C_j] = [R, C_j] = 0 \quad (16b)$$

up to setting

$$X_j = C_j R; \quad (16c)$$

in other words the full relations define a central extension of the 1+1 relation (16a). See [3] for a complete discussion.

Note also that $(I - E) \otimes I$ is a central orthogonal projection onto the subspace where all C^j 's vanish, as well as the X^j 's and R . Indeed, among all possible representations of the κ -Minkowski relations there are the trivial ones, where all the space coordinates are zero; omitting them would be equivalent to removing the time axis through the origin.

It follows that all possible equivalent representations belonging to any $v \in H$ can be obtained from the disintegration of

$$X_{(v)}^\mu = \Lambda^\mu{}_\nu X_{(v(0))}^\nu, \quad (17)$$

where $\Lambda v_{(0)} = v$.

We are now ready to construct the covariant representation; instead of using direct integrals, we simply describe the result: let

$$\mathfrak{H} = L^2(H, dv; \mathfrak{H}_0)$$

be the space of L^2 vector valued functions of H , with values in \mathfrak{H}_0 , where dv is the Lorentz invariant measure on the mass 1 hyperboloid H . Moreover, let $\Lambda(v)$ be a continuous function of H with values in \mathcal{L} , fulfilling $\Lambda(v)v_{(0)} = v$. Then set

$$\begin{aligned} (X^\mu \Psi)(v) &= \Lambda(v)^\mu{}_\nu X_{(v(0))}^\nu \Psi(v), \\ (V^\mu \Psi)(v) &= v^\mu \Psi(v), \\ (U(\Lambda) \Psi)(v) &= \Psi(\Lambda^{-1}v), \end{aligned}$$

where $v \in H, \Lambda \in \mathcal{L}$ and we recall that, for each v , $\Psi(v) \in \mathfrak{H}_0$.

²We recall that if ψ is a common eigenvector for the operators A_1, \dots, A_n , so that $A_j \psi = a_j \psi$, then the n -tuple (a_1, \dots, a_n) is in the joint spectrum of A_1, \dots, A_n , which is a subset of \mathbb{R}^n . The joint spectrum also contains all n -tuples corresponding to possibly generalised common eigenvectors, so that some a_j in a given n -tuple may well belong to the continuous spectrum of the corresponding A_j .

By construction these operators fulfil (3,4). The above representation can be easily shown not to depend on the choice of the map $\Lambda(v)$.

By standard techniques (see again [3]), the universal C*-algebra generated by $f(V, X)$ (as f runs in the admissible symbols) can be shown to be

$$\mathcal{C}_\infty(H) \otimes ((\mathcal{C}(S^2) \otimes \mathcal{K}) \oplus \mathcal{C}_\infty(\mathbb{R})),$$

where \mathcal{C}_∞ means continuous and vanishing at infinity. The picture is as follows: the compact C*-algebra \mathcal{K} arises from the quantization of time and radius, the latter being strictly greater than zero; the commutative factor $\mathcal{C}(S^2)$ describes the classical angle variables. Hence $\mathcal{C}(S^2) \otimes \mathcal{K}$ describes the quantization of the Minkowski spacetime with the time axis through the origin removed; adding $\mathcal{C}_\infty(\mathbb{R})$ restores the missing axis as a classical submanifold which is topologically disjoint from the rest. Finally H appears as additional degrees of freedom, surviving the classical limit as a hidden manifold (extra dimensions).

References

- [1] J. Lukierski, H. Ruegg, A. Nowicki, and V. N. Tolstoy, “ q -deformation of Poincare algebra,” *Phys. Lett.* **B264** (1991) 331–338.
- [2] S. Majid and H. Ruegg, “Bicrossproduct structure of κ -Poincare group and noncommutative geometry,” *Phys. Lett.* **B334** (1994) 348–354, [arXiv:hep-th/9405107](#).
- [3] L. Dabrowski and G. Piacitelli, “Canonical κ -Minkowski Spacetime.” 2010.
- [4] L. Dabrowski and G. Piacitelli, “The κ -Minkowski Spacetime: Trace, Classical Limit and uncertainty Relations,” in Kowalski-Glikman *et. al.* [19]. [arXiv:0909.3215 \[hep-th\]](#).
- [5] S. Doplicher, K. Fredenhagen, and J. E. Roberts, “The Quantum structure of space-time at the Planck scale and quantum fields,” *Commun. Math. Phys.* **172** (1995) 187–220, [arXiv:hep-th/0303037](#).
- [6] J. Lukierski, V. Lyakhovsky, and M. Mozrzymas, “ κ -deformations of $D = 4$ Weyl and conformal symmetries,” *Phys. Lett.* **B538** (2002) 375–384, [arXiv:hep-th/0203182](#).
- [7] J. Kowalski-Glikman, “Introduction to doubly special relativity,” *Lect. Notes Phys.* **669** (2005) 131–159, [arXiv:hep-th/0405273](#).
- [8] G. Piacitelli, “Twisted Covariance as a Non Invariant Restriction of the Fully Covariant DFR Model,” *Commun. Math. Phys.* **295** (2010) 701–729, [arXiv:0902.0575 \[hep-th\]](#).
- [9] G. Piacitelli, “Twisted Covariance and Weyl Quantisation,” in Kowalski-Glikman *et. al.* [19]. [arXiv:0901.3109 \[hep-th\]](#).
- [10] M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu, “On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT,” *Phys. Lett.* **B604** (2004) 98–102, [arXiv:hep-th/0408069](#).
- [11] J. Wess, “Deformed coordinate spaces: Derivatives,” in *Mathematical, Theoretical and Phenomenological Challenges Beyond the Standard Model. Perspectives of the Balcan Collaboration, 29 August – 2 September 2003, Vrnjačka Banja*, G. Djordjevic and J. Wess, eds. Word Scientific, 2005. [arXiv:hep-th/0408080](#).
- [12] M. Chaichian, P. Prešnajder, and A. Tureanu, “New concept of relativistic invariance in NC space-time: Twisted Poincare symmetry and its implications,” *Phys. Rev. Lett.* **94** (2005) 151602, [arXiv:hep-th/0409096](#).
- [13] P. Aschieri, M. Dimitrijevic, P. Kulish, F. Lizzi, and J. Wess, *Noncommutative spacetimes: Symmetries in noncommutative geometry and field theory*. Springer, 2009.

- [14] J. Neumann, von, “Über die eindeutigkeit der Schrödingerschen operatoren,” *Math. Annalen* **104** (1931) 570–578, [GDZ:purl?GDZPPN002274701](#).
- [15] A. Agostini, F. Lizzi, and A. Zampini, “Generalized Weyl systems and kappa-Minkowski space,” *Mod. Phys. Lett. A* **17** (2002) 2105–2126, [arXiv:hep-th/0209174](#).
- [16] P. Kosinski, J. Lukierski, and P. Maslanka, “Local field theory on kappa-Minkowski space, star products and noncommutative translations,” *Czech. J. Phys.* **50** (2000) 1283–1290, [arXiv:hep-th/0009120](#).
- [17] M. Dimitrijevic, L. Jonke, L. Möller, E. Tsouchnika, J. Wess, and M. Wohlgenannt, “Deformed field theory on kappa-spacetime,” *Eur. Phys. J. C* **31** (2003) 129–138, [arXiv:hep-th/0307149](#).
- [18] S. Doplicher, K. Fredenhagen, and J. E. Roberts, “Space-time quantization induced by classical gravity,” *Phys. Lett. B* **331** (1994) 39–44.
- [19] J. Kowalski-Glikman, R. Durka, and M. Szczachor, eds., *The Planck Scale. Proceedings of the XXV Max Born Symposium, Wrocław, 29 June – 3 July 2009*, no. 1196 in Conference Proceedings. AIP, Melville, New York, 2010.